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Technical Notes

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Two-Dimensional Analysis of Functionally Graded Beams

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I. Introduction

FUNCTIONALLY graded materials (FGMs) in which the volume fractions of two or more material constituents are designed to vary continuously as a function of position along certain direction(s) have been developed and studied in last two decades. The advantage of FGMs is that no distinct internal boundaries exist and failures from interfacial stress concentrations developed in conventional components can be avoided. The gradual change of material properties can be tailored to different applications and working environments.

The literature on the response of FGM beam to mechanical and other loadings is limited. Shi and his coworkers studied the response of functionally graded piezoelectric material (FGPM) beams [1–3]. But in their analysis, only one or two material parameters were assumed to vary in the form of finite power series along the thickness direction while other parameters kept constant. Sankar and his coworkers [4–7] developed analytical methods for the thermomechanical and contact analysis of FGM beams and also for sandwich beams with FGM cores. In their studies the thermomechanical properties of the FGM were assumed to vary through the thickness in an exponential fashion. Zhu and Sankar [8] studied a FGM beam whose Young's modulus was given by a polynomial in the thickness coordinate. A new beam element based on the first-order shear deformation theory was developed to study the thermoelastic behavior of FGM beam structures by Chakraborty et al. [9]. In those papers, both exponential and power variations of material property distribution were employed. As far as we know, all the available solutions are valid for some specific material distribution and no general solution exists that is adaptive to arbitrary material properties variation for FGMs.

The objective of this work is to present a general solution of a functionally graded beam with arbitrary graded variations of material property based on two-dimensional theory of elasticity.

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II. Formulation

Consider a FGM beam of arbitrary composition gradient through the thickness, as shown in Fig. 1. The thickness, length, and width of the beam are denoted, respectively, by h, l, and b, A Cartesian coordinate system is introduced whose x–y plane coincides with the midplane of the beam and the z-axis located along the thickness direction. The beam is assumed to be in a state of plane stress or plane strain in which it is loaded by forces applied to the boundary, parallel to the x–z plane and distributed uniformly over the y-direction.

In the absence of body forces the differential equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{zx}}{\partial z}=0 \qquad \frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \sigma_z}{\partial z}=0 \tag{1}$$

where σ_x, σ_z, τ_{zx} are stress components. Assuming that the material is orthotropic at every point and the principal material directions coincide with the x and z axes, the constitutive relations are given as

$$\varepsilon_x = s_{11}\sigma_x + s_{13}\sigma_z \qquad \varepsilon_z = s_{13}\sigma_x + s_{33}\sigma_z \qquad \gamma_{zx} = s_{44}\tau_{zx} \tag{2}$$

where s_{11}, s_{33}, s_{13}, s_{44} are elastic compliances for plane stress or plane strain cases, and ε_x, ε_z, γ_{zx} are strain components that are related to the displacements components u, w by the following relations:

$$\varepsilon_x=\frac{\partial u}{\partial x} \qquad \varepsilon_z=\frac{\partial w}{\partial z} \qquad \gamma_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \tag{3}$$

The strain components also satisfy the strain compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial z^2}+\frac{\partial^2 \varepsilon_z}{\partial x^2}-\frac{\partial^2 \gamma_{zx}}{\partial z \partial x}=0 \tag{4}$$

In contrast to a homogeneous material, the elastic moduli s_{11}, s_{33}, s_{13}, s_{44} for a FGM are now functions of coordinates. For most real applications, the material properties are designed to vary continuously only in one direction. In the present paper, we assume that the elastic moduli vary only along z-direction, i.e.,

$$s_{ij}=s_{ij}(z) \tag{5}$$

where s_{ij} represents s_{11}, s_{33}, s_{13}, s_{44}.

To satisfy the equations of equilibrium (1), Airy stress function $U(x,z)$ is introduced such that

$$\sigma_x=\frac{\partial^2 U}{\partial z^2} \qquad \sigma_z=\frac{\partial^2 U}{\partial x^2} \qquad \tau_{zx}=-\frac{\partial^2 U}{\partial z \partial x} \tag{6}$$

Substituting Eqs. (2) and (6) into Eq. (4), the governing equation for Airy stress function $U(x,z)$ can be obtained as

$$\frac{\partial^2}{\partial z^2}\left(s_{11}\frac{\partial^2 U}{\partial z^2}+s_{13}\frac{\partial^2 U}{\partial x^2}\right)+\frac{\partial}{\partial z}\left(s_{44}\frac{\partial^3 U}{\partial x^2 \partial z}\right)+s_{13}\frac{\partial^4 U}{\partial x^2 \partial z^2} + s_{33}\frac{\partial^4 U}{\partial x^4}=0 \tag{7}$$

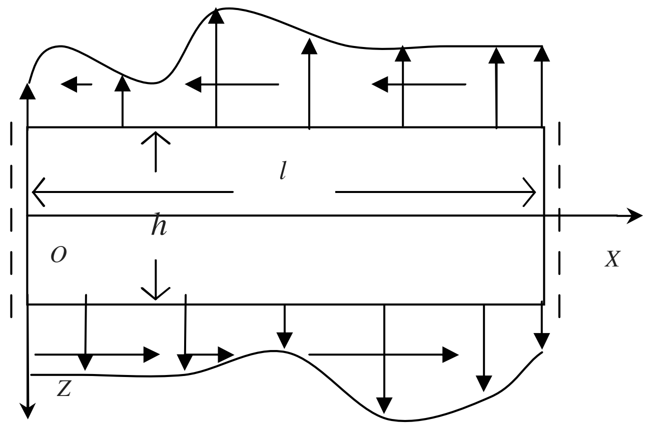

Fig. 1 Schematic of a functionally graded beam.

III. Solution

If we assume the following form of Airy stress function

$$U = \sum_{i=0}^{n} x^i f_i(z) \tag{8}$$

then we have

$$\sigma_x = \sum_{i=0}^{n} x^i \frac{\mathrm{d}^2 f_i(z)}{\mathrm{d}z^2} \qquad \tau_{zx} = -\sum_{i=1}^{n} i x^{i-1} \frac{\mathrm{d} f_i(z)}{\mathrm{d}z}$$
$$\sigma_z = \sum_{i=2}^{n} i(i-1) x^{i-2} f_i(z) \tag{9}$$

Substituting Eq. (8) into Eq. (7), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(s_{11} \frac{\mathrm{d}^2 f_i(z)}{\mathrm{d}z^2}\right) = F_i^0(z) \qquad (i = 0, 1, \ldots, n) \tag{10}$$

where

$$F_i^0(z) = \begin{cases} 0 & (i = n-1, n) \\ -(i+1)(i+2)\left[\frac{\mathrm{d}^2 R_{i+2}^{13}(z)}{\mathrm{d}z^2} + \frac{\mathrm{d}S_{i+2}^{44}(z)}{\mathrm{d}z} + T_{i+2}^{13}\right] & (i = n-3, n-2) \\ -(i+1)(i+2)\left[\frac{\mathrm{d}^2 R_{i+2}^{13}(z)}{\mathrm{d}z^2} + \frac{\mathrm{d}S_{i+2}^{44}(z)}{\mathrm{d}z} + T_{i+2}^{13} + (i+3)(i+4)R_{i+4}^{33}\right] & (i = 0, 1, \ldots, n-4) \end{cases} \tag{11}$$

with the following notations defined as

$$R_i^{kl}(z) = s_{kl}(z) f_i(z) \qquad S_i^{kl}(z) = s_{kl}(z) \frac{\mathrm{d} f_i(z)}{\mathrm{d}z}$$
$$T_i^{kl}(z) = s_{kl}(z) \frac{\mathrm{d}^2 f_i(z)}{\mathrm{d}z^2} \qquad (kl = 11, 33, 13, 44) \tag{12}$$

Eq. (10) gives a recurrence relation for $f_i(z)$. It is easy to find the general solution for $f_n(z)$ and $f_{n-1}(z)$ because $F_n^0(z) = F_{n-1}^0(z) = 0$. Then $f_i(z)$ $(i = n-2, \ldots, 1, 0)$ can be solved one by one using the obtained solution for $f_n(z)$ and $f_{n-1}(z)$. Hence, the solution of $f_i(z)$ $(i = 0, 1, \ldots, n)$ is written in a general form as

$$f_1(z) = F_i^4(z) + A_i \hat{H}_1(z) + B_i \hat{H}_0(z) + C_i z + D_i \tag{13}$$

where A_i, B_i, C_i, D_i are unknown constants to be determined, and

$$F_i^1(z) = \int_0^z F_i^0(z)\,\mathrm{d}z \qquad F_i^2(z) = \int_0^z F_i^1(z)\,\mathrm{d}z$$
$$F_i^3(z) = \int_0^z \frac{F_i^2(z)}{s_{11}(z)}\,\mathrm{d}z \qquad F_i^4(z) = \int_0^z F_i^3(z)\,\mathrm{d}z$$
$$F_{n-1}^1(z) = F_n^1(z) = F_{n-1}^2(z) = F_n^2(z) = F_{n-1}^3(z) = F_n^3(z) = F_{n-1}^4(z) = F_n^4(z) = 0$$
$$H_k(z) = \int_0^z \frac{z^k\,\mathrm{d}z}{s_{11}(z)} \qquad \hat{H}_k(z) = \int_0^z H_k(z)\,\mathrm{d}z = z H_k(z)$$
$$(k = 0, 1) \tag{14}$$

Note that C_0, D_0, and D_1 can be ignored because they make no contribution to the stress field.

Accordingly, the displacements can be obtained as

$$u = \sum_{i=0}^{n} \frac{x^{i+1}}{i+1} T_i^{11}(z) + \sum_{i=2}^{n} i x^{i-1} R_i^{13}(z) - \bar{T}_1^{13}(z) - 6\bar{R}_3^{33}(z) - \hat{S}_1^{44}(z) - az + c$$
$$w = \sum_{i=0}^{n} x^i \hat{T}_i^{13}(z) + \sum_{i=2}^{n} i(i-1) x^{i-2} \hat{R}_i^{33}(z) - \sum_{i=0}^{n} \frac{x^{i+2}}{(i+1)(i+2)} A_i + ax + d \tag{15}$$

where a, c, d are unknown constants related to the rigid motions of the beam, and

$$\hat{R}_i^{kl}(z) = \int_0^z R_i^{kl}(z)\,\mathrm{d}z \qquad \hat{S}_i^{kl}(z) = \int_0^z S_i^{kl}(z)\,\mathrm{d}z$$
$$\hat{T}_i^{kl}(z) = \int_0^z T_i^{kl}(z)\,\mathrm{d}z \qquad \bar{R}_i^{kl}(z) = \int_0^z \hat{R}_i^{kl}(z)\,\mathrm{d}z$$
$$\bar{S}_i^{kl}(z) = \int_0^z \hat{S}_i^{kl}(z)\,\mathrm{d}z \qquad \bar{T}_i^{kl}(z) = \int_0^z \hat{T}_i^{kl}(z)\,\mathrm{d}z$$
$$(kl = 11, 33, 13, 44) \tag{16}$$

Up to now, the basic formulation has been established except that $4n+4$ unknown constants, $a, c, d, A_0, B_0, A_1, B_1, C_1$, and A_i, B_i, C_i, D_i $(i = 2, \ldots, n)$, need to be determined from boundary conditions of the beam.

Accordingly, the concentrated normal force N_0, concentrated shear force P_0, and concentrated moment M_0 at the left end $(x = 0)$ of the beam can be calculated as follows:

$$N_0 = b\int_{-h/2}^{h/2} \sigma_x|_{x=0}\,\mathrm{d}z \qquad M_0 = b\int_{-h/2}^{h/2} \sigma_x|_{x=0} z\,\mathrm{d}z$$
$$P_0 = b\int_{-h/2}^{h/2} \tau_{zx}|_{x=0}\,\mathrm{d}z \tag{17}$$

Their counterparts at right end ($x = l$), N_l, P_l, and M_l, are given by

$$N_l = b\int_{-h/2}^{h/2} \sigma_x|_{x=l}\,dz \qquad M_l = b\int_{-h/2}^{h/2} \sigma_x|_{x=l} z\,dz \qquad P_l = b\int_{-h/2}^{h/2} \tau_{zx}|_{x=l}\,dz \tag{18}$$

If the top and bottom surfaces of the beam are subjected to normal and shear tractions of polynomial form, as follows

$$\sigma_z|_{z=h/2} = \sum_{i=0}^{n-2} a_i x^i \qquad \tau_{zx}|_{z=h/2} = \sum_{i=0}^{n-1} b_i x^i \tag{19}$$

$$\sigma_z|_{z=-\frac{h}{2}} = \sum_{i=0}^{n-2} c_i x^i \qquad \tau_{zx}|_{z=-\frac{h}{2}} = \sum_{i=0}^{n-1} d_i x^i \tag{20}$$

then we have

$$\begin{aligned} &f_i\left(\tfrac{h}{2}\right) = \tfrac{a_{i-2}}{i(i-1)} \qquad f_i\left(-\tfrac{h}{2}\right) = \tfrac{c_{i-2}}{i(i-1)} \qquad (i = 2,\ldots,n) \\ &\left.\tfrac{df_i}{dz}\right|_{z=h/2} = -\tfrac{b_{i-1}}{i} \qquad \left.\tfrac{df_i}{dz}\right|_{z=-h/2} = -\tfrac{d_{i-1}}{i} \qquad (i = 1,\ldots,n) \end{aligned} \tag{21}$$

or

$$\begin{aligned} &A_i\hat{H}_1\left(\tfrac{h}{2}\right) + B_i\hat{H}_0\left(\tfrac{h}{2}\right) + C_i\left(\tfrac{h}{2}\right) + D_i + F_i^4\left(\tfrac{h}{2}\right) = \tfrac{a_{i-2}}{i(i-1)} \quad (i = 2,\ldots,n) \\ &A_i\hat{H}_1\left(-\tfrac{h}{2}\right) + B_i\hat{H}_0\left(-\tfrac{h}{2}\right) - C_i\left(\tfrac{h}{2}\right) + D_i + F_i^4\left(-\tfrac{h}{2}\right) = \tfrac{c_{i-2}}{i(i-1)} \quad (i = 2,\ldots,n) \\ &A_iH_1\left(\tfrac{h}{2}\right) + B_iH_0\left(-\tfrac{h}{2}\right) + C_i + F_i^3\left(\tfrac{h}{2}\right) = -\tfrac{b_{i-1}}{i} \quad (i = 1,\ldots,n) \\ &A_iH_1\left(-\tfrac{h}{2}\right) + B_iH_0\left(-\tfrac{h}{2}\right) + C_i + F_i^3\left(-\tfrac{h}{2}\right) = -\tfrac{d_{i-1}}{i} \quad (i = 1,\ldots,n) \end{aligned} \tag{22}$$

Because $F_i^4(\frac{h}{2})$, $F_i^4(-\frac{h}{2})$, $F_i^3(\frac{h}{2})$, $F_i^3(-\frac{h}{2})$ are all linear functions of A_j, B_j, C_j, D_j $(j = i+2,\ldots,n)$, (19) constitutes $4n-2$ independent algebraic equations for unknowns A_1, B_1, C_1, and A_i, B_i, C_i, D_i $(i = 2,\ldots,n)$. To uniquely determine all $4n+4$ unknowns, other six equations are needed, which are obtained from end boundary conditions, as follows.

A. Cantilever Beams

For a cantilever FGM beam clamped at one end ($x = l$) and subjected to a concentrated normal force $\bar{N}_0$, a concentrated shear force $\bar{P}_0$, and a concentrated moment $\bar{M}_0$ at another end ($x = 0$), the end boundary conditions are given as

$$N_0 = \bar{N}_0 \qquad P_0 = \bar{P}_0 \qquad M_0 = \bar{M}_0 \quad \text{at } x = 0 \tag{23}$$

$$u = w = \frac{\partial w}{\partial x} = 0 \quad \text{at } x = l, z = 0 \tag{24}$$

Substituting (9) and (17) into (23), we obtain

$$\begin{aligned} &A_0\left[H_1\left(\frac{h}{2}\right) - H_1\left(-\frac{h}{2}\right)\right] + B_0\left[H_0\left(\frac{h}{2}\right) - H_0\left(-\frac{h}{2}\right)\right] + \left[F_0^3\left(\frac{h}{2}\right) - F_0^3\left(-\frac{h}{2}\right)\right] = \frac{\bar{N}_0}{b} \\ &A_0\left[H_2\left(\frac{h}{2}\right) - H_2\left(-\frac{h}{2}\right)\right] + B_0\left[H_1\left(\frac{h}{2}\right) - H_1\left(-\frac{h}{2}\right)\right] \\ &\quad + \frac{h}{2}\left[F_0^3\left(\frac{h}{2}\right) + F_0^3\left(-\frac{h}{2}\right)\right] - F_0^4\left(\frac{h}{2}\right) + F_0^4\left(-\frac{h}{2}\right) = \frac{\bar{M}_0}{b} \\ &A_1\left[\hat{H}_1\left(\frac{h}{2}\right) - \hat{H}_1\left(-\frac{h}{2}\right)\right] + B_1\left[\hat{H}_0\left(\frac{h}{2}\right) - \hat{H}_0\left(-\frac{h}{2}\right)\right] + C_1h \\ &\quad + F_1^4\left(\frac{h}{2}\right) - F_1^4\left(-\frac{h}{2}\right) = -\frac{\bar{P}_0}{b} \end{aligned} \tag{25}$$

Obviously, the $4n+1$ unknowns A_0, B_0, A_1, B_1, C_1, and A_i, B_i, C_i, D_i $(i = 2,\ldots,n)$ are uniquely solved from $4n+1$ independent algebraic equations described by (22) and (25). Substituting (15) into (24), we obtain

$$\begin{aligned} a &= \sum_{i=0}^{n} \frac{1}{i+1} l^{i+1} A_i \\ c &= -\sum_{i=0}^{n} \frac{1}{i+1} l^{i+1} B_i - \sum_{i=2}^{n} i l^{i-1} s_{kl}(0) D_i \\ d &= -\sum_{i=0}^{n} \frac{1}{i+2} l^{i+2} A_i \end{aligned} \tag{26}$$

B. Simply Supported Beams

For a FGM beam simply supported at both ends ($x = 0$ and $x = l$) and subjected to a concentrated normal force $\bar{N}_0$, a concentrated moment $\bar{M}_0$ at its left end ($x = 0$), and a concentrated moment $\bar{M}_l$ at its right end, the end boundary conditions are written as

$$\begin{aligned} &N_0 = \bar{N}_0 \qquad M_0 = \bar{M}_0 \quad \text{at } x = 0 \\ &M_0 = \bar{M}_l \qquad \text{at } x = l \end{aligned} \tag{27}$$

$$\begin{aligned} w &= 0 \qquad \text{at } x = 0,\ z = 0 \\ u = w &= 0 \qquad \text{at } x = l,\ z = 0 \end{aligned} \tag{28}$$

Substituting (9), (15), (17), and (18) into (27), we obtain the same equations as those in (25), with

$$\frac{\bar{P}_0}{b} = \frac{\bar{M}_l - \bar{M}_0}{bl} + \sum_{i=0}^{n-2}(a_i - c_i)\frac{l^{i+1}}{(i+1)(i+2)} + \frac{h}{2}\sum_{i=0}^{n-1}(b_i + d_i)\frac{l^i}{i+1} \tag{29}$$

Similarly, the $4n+1$ unknowns A_0, B_0, A_1, B_1, C_1, and A_i, B_i, C_i, D_i $(i = 2,\ldots,n)$ can be determined from (22) and (25). From (28) we can determine

$$\begin{aligned} a &= \sum_{i=0}^{n} \frac{l^{i+1}}{(i+1)(i+2)} A_i \\ c &= -\sum_{i=0}^{n} \frac{l^{i+1}}{i+1} B_i - \sum_{i=2}^{n} i l^{i-1} s_{13}(0) D_i \qquad d = 0 \end{aligned} \tag{30}$$

C. Rigidly Clamped Beams

For a FGM beam clamped at both ends ($x = 0$ and $x = l$), the end boundary conditions are given as

$$u = w = \frac{\partial w}{\partial x} = 0 \qquad \text{at } x = 0,\ z = 0 \quad \text{and} \quad x = l,\ z = 0 \quad (31)$$

Substituting (15) into (31), we obtain

$$a = c = d = 0 \quad (32)$$

and

$$\sum_{i=0}^{n} \frac{l^{i+1}}{i+1} A_i = 0 \qquad \sum_{i=0}^{n} \frac{l^{i+2}}{i+2} A_i = 0$$
$$\sum_{i=0}^{n} \frac{l^{i+1}}{i+1} B_i + \sum_{i=2}^{n} i l^{i-1} s_{13}(0) D_i = 0 \quad (33)$$

From (22) and (33), the $4n + 1$ unknowns A_0, B_0, A_1, B_1, C_1, and A_i, B_i, C_i, D_i ($i = 2, \ldots, n$) can be uniquely obtained.

IV. Examples

In this section we list the results for two specific examples.

Example 1: For a cantilever FGM beam only subjected to a concentrated shear force $\bar{P}_0$ at the free end ($x = 0$), the stresses and displacements are derived as

$$\sigma_x = \frac{1}{s_{11}(z)} x(A_1 z + B_1) \qquad \sigma_z = 0$$
$$\tau_{zx} = -A_1 H_1(z) - B_1 H_0(z) - C_1 \quad (34)$$

$$u = \frac{1}{2} x^2 (A_1 z + B_1) - A_1 \hat{W}_1(z) - B_1 \hat{W}_0(z) - A_1 \Phi_1(z) - B_1 \Phi_0(z) - C_1 \hat{s}_{44}(z) - A_1 \frac{l^2}{2} z - B_1 \frac{l^2}{2}$$
$$w = x[A_1 W_1(z) + B_1 W_0(z)] - A_1 \frac{x^3}{6} + A_1 \frac{l^2}{2} x - A_1 \frac{l^3}{3} \quad (35)$$

where

$$W_i(z) = \int_0^z \frac{s_{13}(z)}{s_{11}(z)} z^i \, \mathrm{d}z \qquad (i = 0, 1)$$
$$\Phi_0(z) = \int_0^z H_0(z) s_{44}(z) \, \mathrm{d}z \qquad \Phi_1(z) = \int_0^z H_1(z) s_{44}(z) \, \mathrm{d}z$$
$$\hat{s}_{44}(z) = \int_0^z s_{44}(z) \, \mathrm{d}z \qquad \hat{W}_i(z) = \int_0^z W_i(z) \, \mathrm{d}z \qquad (i = 0, 1) \quad (36)$$

and

$$A_1 = -\frac{\bar{P}_0}{bK} \left[H_0\left(\frac{h}{2}\right) - H_0\left(-\frac{h}{2}\right) \right]$$
$$B_1 = \frac{\bar{P}_0}{bK} \left[H_1\left(\frac{h}{2}\right) - H_1\left(-\frac{h}{2}\right) \right]$$
$$C_1 = -\frac{\bar{P}_0}{bK} \left[H_1\left(\frac{h}{2}\right) H_0\left(-\frac{h}{2}\right) - H_1\left(-\frac{h}{2}\right) H_0\left(\frac{h}{2}\right) \right] \quad (37)$$

with

$$K = \left[\hat{H}_1\left(\frac{h}{2}\right) - \hat{H}_1\left(-\frac{h}{2}\right) \right] \left[H_0\left(-\frac{h}{2}\right) - H_0\left(\frac{h}{2}\right) \right] - \left[\hat{H}_0\left(\frac{h}{2}\right) - \hat{H}_0\left(-\frac{h}{2}\right) \right] \left[H_1\left(\frac{h}{2}\right) - H_1\left(-\frac{h}{2}\right) \right] + h \left[H_1\left(\frac{h}{2}\right) H_0\left(-\frac{h}{2}\right) - H_1\left(-\frac{h}{2}\right) H_0\left(\frac{h}{2}\right) \right] \quad (38)$$

For a homogeneous and isotropic beam in the state of plane stress ($b \ll l$) with Young's modulus E and Poisson's ratio ν, we have

$$s_{11} = s_{33} = \frac{1}{E} \qquad s_{13} = -\frac{\nu}{E} \qquad s_{44} = \frac{1}{G} = \frac{2(1+\nu)}{E} \quad (39)$$

and

$$\sigma_x = \frac{\bar{P}_0}{I} xz \qquad \sigma_z = 0 \qquad \tau_{zx} = \frac{\bar{P}_0}{2I} \left(\frac{h^2}{4} - z^2 \right) \quad (40)$$

$$u = \frac{\bar{P}_0}{2EI} x^2 z - \frac{\bar{P}_0}{6I} \left(\frac{1}{G} - \frac{\upsilon}{E} \right) z^3 + \frac{\bar{P}_0}{2I} \left(\frac{h^2}{4G} - \frac{l^2}{E} \right) z$$
$$w = -\frac{\upsilon \bar{P}_0}{2EI} xz^2 - \frac{\bar{P}_0}{6EI} x^3 + \frac{\bar{P}_0 l^2}{2EI} x - \frac{\bar{P}_0 l^3}{3EI} \quad (41)$$

where $I = bh^3/12$. The preceding results (40) and (41) are in conformity with the classical solution of a homogeneous and isotropic cantilever beam [10] except that the shear force is in an opposite direction.

Example 2: For a rigidly clamped FGM beam only subjected to a uniform pressure q on its upper surface, the stresses and displacements are obtained as

$$\sigma_x = \frac{x^2}{s_{11}(z)} (A_2 z + B_2) - 2 \frac{s_{13}(z)}{s_{11}(z)} [A_2 \hat{H}_1(z) + B_2 \hat{H}_0(z) + C_2 z + D_2] - \frac{2}{s_{11}(z)} [A_2 \hat{W}_1(z) + B_2 \hat{W}_0(z) + A_2 \Phi_1(z) + B_2 \Phi_0(z) + C_2 \hat{s}_{44}(z)] + \frac{1}{s_{11}(z)} (A_0 z + B_0)$$
$$\sigma_z = 2[A_2 H_1(z) + B_2 \hat{H}_0(z) + C_2 z + D_2]$$
$$\tau_{zx} = -2x[A_2 H_1(z) + B_2 H_0(z) + C_2] \quad (42)$$

$$u = \frac{x^3}{3} (A_2 z + B_2) - 2x[A_2 \hat{W}_1(z) + B_2 \hat{W}_0(z) + A_2 \Phi_1(z) + B_2 \Phi_0(z) + C_2 \hat{s}_{44}(z)] + x(A_0 z + B_0)$$
$$w = x^2 [A_2 W_1(z) + B_2 W_0(z)] - 2[A_2 \hat{V}_1(z) + B_2 \hat{V}_0(z) + C_2 \Lambda_1(z) + D_2 \Lambda_0(z)] - 2[A_2 \bar{\Pi}_1(z) + B_2 \bar{\Pi}_0(z) + A_2 \Psi_1(z) + B_2 \Psi_0(z) + C_2 \hat{\Gamma}_{44}(z)] + A_0 W_1(z) + B_0 W_0(z) - \frac{x^4}{12} A_2 - \frac{x^2}{2} A_0 \quad (43)$$

where $W_1(z)$, $W_0(z)$, $\hat{W}_1(z)$, $\hat{W}_0(z)$, $\Phi_0(z)$, $\Phi_1(z)$, $\hat{s}_{44}(z)$ have been given in (36) and

$$\Psi_k(z)=\int_0^z \frac{s_{13}(z)}{s_{11}(z)}\Phi_k(z)\,\mathrm{d}z \qquad \bar{\Pi}_k(z)=\int_0^z \frac{s_{13}(z)}{s_{11}(z)}\hat{W}_k(z)\,\mathrm{d}z$$
$$\hat{\Gamma}_{44}(z)=\int_0^z \frac{s_{13}(z)}{s_{11}(z)}\hat{s}_{44}(z)\,\mathrm{d}z$$
$$\hat{V}_k(z)=\int_0^z \left[\frac{s_{13}^2(z)}{s_{11}(z)}-s_{13}(z)\right]\hat{H}_k(z)\,\mathrm{d}z$$
$$\Lambda_k(z)=\int_0^z \left[\frac{s_{13}^2(z)}{s_{11}(z)}-s_{13}(z)\right]z^k\,\mathrm{d}z \qquad (k=0,1) \tag{44}$$

$$A_2=\frac{q}{2K}\left[H_0\left(\frac{h}{2}\right)-H_0\left(-\frac{h}{2}\right)\right]$$
$$B_2=-\frac{q}{2K}\left[H_1\left(\frac{h}{2}\right)-H_1\left(-\frac{h}{2}\right)\right]$$
$$C_2=\frac{q}{2K}\left[H_1\left(\frac{h}{2}\right)H_0\left(-\frac{h}{2}\right)-H_1\left(-\frac{h}{2}\right)H_0\left(\frac{h}{2}\right)\right]$$
$$D_2=-\frac{q}{2K}\left\{\left[H_0\left(\frac{h}{2}\right)-H_0\left(-\frac{h}{2}\right)\right]\hat{H}_1\left(\frac{h}{2}\right)-\left[H_1\left(\frac{h}{2}\right)-H_1\left(-\frac{h}{2}\right)\right]\hat{H}_0\left(\frac{h}{2}\right)\right\}-\frac{q}{2K}\frac{h}{2}\left[H_1\left(\frac{h}{2}\right)H_0\left(-\frac{h}{2}\right)-H_1\left(-\frac{h}{2}\right)H_0\left(\frac{h}{2}\right)\right]$$
$$A_1=-lA_2 \qquad A_0=\frac{l^2}{6}A_2 \qquad B_0=-\frac{1}{2}B_1-\frac{l^2}{3}B_2-2s_{13}(0)D_2$$
$$B_1=-\frac{H_1(h/2)-H_1(-h/2)}{H_0(h/2)-H_0(-h/2)}A_1$$
$$C_1=\frac{H_1(h/2)H_0(-h/2)-H_1(-h/2)H_0(h/2)}{H_0(h/2)-H_0(-h/2)}A_1 \tag{45}$$

where K is given in (38).

V. Conclusions

By means of the semi-inverse method, plane elasticity solutions are obtained for orthotropic functionally graded beams with arbitrary elastic moduli variations along the thickness direction under different end boundary conditions. Although the solution technique is only valid for the case where the beam is subjected to normal and shear tractions of polynomial form on the top and bottom surfaces, the obtained solutions are useful for many engineering applications and they can serve as a basis for establishing simplified FGM beam theories or as a benchmark result to assess other approximate methodologies.

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